ON HYPERSONIC FLOWS OF VISCOUS HEAT-CONDUCTING GASES

(O GIPERZVUKOVYKH TECHENIAKH VIAZKOGO TEPLOPROVODNOGO GAZA)

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In studies of flow fields around bodies, moving with large supersonic velocities, the influence of viscosity and heat-conductivity may be significant. In some flight regimes, this influence manifests itself as an interaction between the boundary layer and the outer inviscid flow [1]. At still higher flight speeds it may force the investigation of the whole flow field on the basis of the complete equations of motion of a viscous heat-conducting gas. In the present paper, several flows of a viscous heat-conducting gas are studied on the basis of the full Navier-Stokes equations for the special case of an infinite Mach number. If the density and the speed of the undisturbed flow are kept constant as the Mach number grows without limit, so that the temperature, pressure, and enthalpy approach zero, then, as is well known [2], a limiting flow field is reached, which is independent of Mach number. This result represents a generalization to the viscous heat-conducting gas of the known gasdynamical independence principle of flows at very high speeds [3]. In this sense, a flow of a viscous heat-conducting gas at very high speeds can be viewed as a flow with vanishing temperature in the undisturbed region.

At the outset we note the main characteristic of such a flow, namely, that in this case we always have a surface (front) which divides the region of the disturbed flow from that filled with undisturbed gas, rather than a configuration in which the disturbances die out asymptotically at infinity. The explanation lies in the fact that the viscosity as well as the heat-conductivity of the gas are such functions of temperature that they decrease and vanish as the temperature decreases and vanishes.

First we study some general properties of steady hypersonic flows around bodies downstream of strong shock waves. Then we investigate some self-similar unsteady gas flows in which the self-similar character is determined by their special state, namely, by the vanishing temperature in the undisturbed region. Formally, this behavior is connected with the fact that the number of governing parameters of the problem is reduced [4]. Throughout, the gas is assumed to be perfect, with constant specific heats, constant Prandtl number, and a constant power of temperature (or enthalpy) in the expression for viscosity.

1. Structure of the strong shock wave. The simplest type of flow of a viscous heat-conducting gas appears to be that of the uniform steady flow, homogeneous at infinity. Here belongs the well-known flow in the shock wave which propagates into a gas at rest [5]. Let us investigate this flow in the case of an infinite Mach number so that we may clarify the nature of the flow in the neighborhood of the front and obtain several important relationships. Let the speed in the undisturbed flow be equal to V_∞ and the density to ρ_∞, while the pressure p_∞, the temperature T_∞, and the enthalpy h_∞ are equal to zero. For the sake of generality, we take the angle α between the normal to the front (Fig. 1) and the velocity vector V_∞ to be nonzero (oblique shock). We choose the Cartesian x-axis normal to the shock front. We denote the velocity components as u, v, w, and the pressure, density and enthalpy as p, ρ and h, respectively. We introduce the viscosity dependence in the form

$$\mu = Ch^n \tag{1.1}$$

where C is a constant. From the dimensional parameters ρ_∞, V_∞ and C which characterize the problem, we can construct a length l

$$l = \frac{C}{\rho_\infty} V_\infty^{2n-1} \tag{1.2}$$

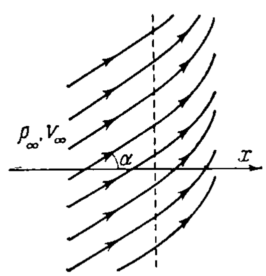

Fig. 1.

(Its physical meaning will be considered later.) Let us now introduce the dimensionless independent variable x° and dimensionless unknown variables

$$x^\circ = \frac{x}{l\cos^{2n-1}\alpha}, \qquad u^\circ = \frac{u}{V_\infty \cos\alpha}, \qquad v^\circ = \frac{v}{V_\infty \sin\alpha}$$
$$p^\circ = \frac{p}{\rho_\infty V_\infty^2 \cos^2\alpha}, \qquad \rho^\circ = \frac{\rho}{\rho_\infty}, \qquad h^\circ = \frac{h}{V_\infty^2 \cos^2\alpha} \tag{1.3}$$

The equations of a steady uniform flow can be represented in terms of these variables as follows:

$$\frac{d\rho^\circ u^\circ}{dx^\circ} = 0, \qquad \rho^\circ u^\circ \frac{du^\circ}{dx^\circ} + \frac{dp^\circ}{dx^\circ} = \frac{4}{3}\frac{d}{dx^\circ}\left(h^{\circ n}\frac{du^\circ}{dx^\circ}\right)$$

$$\rho^\circ u^\circ \frac{dh^\circ}{dx^\circ} = u^\circ \frac{dp^\circ}{dx^\circ} + \frac{1}{\sigma}\frac{d}{dx^\circ}\left(h^{\circ n}\frac{dh^\circ}{dx^\circ}\right) + \frac{4}{3}h^{\circ n}\left(\frac{du^\circ}{dx^\circ}\right)^2, \qquad p^\circ = \frac{\gamma-1}{\gamma}\rho^\circ h^\circ \tag{1.4}$$

Here σ is the Prandtl number and $\gamma = C_p/C_v$, the ratio of the specific heats of the gas. The boundary conditions of this problem read:

$$u^\circ = \rho^\circ = 1, \qquad p^\circ = h^\circ = 0 \quad \text{for} \quad x^\circ \to -\infty \tag{1.5}$$

and the solution is bounded as $x^\circ \to +\infty$.

The lateral velocity in the whole flow field is constant, $v^\circ = 1$. We note that Equation (1.4) and the boundary conditions (1.5), expressed in dimensionless variables, do not contain the parameter α. This fact can be viewed as a law of similarity of flows in shock waves of large intensity.

The system (1.4), as is known [6], can be fully integrated when $\sigma = 3/4$ and $n = 1$. In this case, the particular solution satisfying (1.5) has the form (see Appendix A)

$$\begin{gathered} x^\circ = \frac{2}{3}(1+\varepsilon)\left[(1+\varepsilon)(1-u^\circ) + \varepsilon(1+\varepsilon)\ln\frac{1-\varepsilon}{u^\circ-\varepsilon} + \frac{1-u^{\circ 2}}{2}\right] \\ \rho^\circ = \frac{1}{u^\circ}, \qquad h^\circ = \frac{1-u^{\circ 2}}{2}, \qquad p^\circ = \frac{\gamma-1}{\gamma}\rho^\circ h^\circ \qquad \left(\varepsilon = \frac{\gamma-1}{\gamma+1}\right) \end{gathered} \tag{1.6}$$

This solution is displayed in Fig. 2, where

$$U = u / V_\infty \cos\alpha, \qquad X = x / l \cos\alpha$$

Inspection of the relationships discloses that the disturbed region is separated from the uniform free stream by a surface which has been made to coincide with $x = 0$ by a choice of the unessential arbitrary constant of integration. This surface, in the present case $n = 1$, is also the surface of discontinuity of the derivatives, which take on non-zero values at $x = +0$.

In the more general case of an arbitrary positive value of n, the unknown functions can be represented in the neighborhood of the front ($x \to 0$) as follows (see Appendix A):

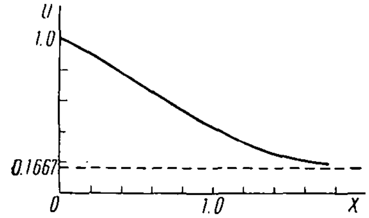

Fig. 2.

$$\begin{gathered} u^\circ \approx 1 - \left(\frac{3n}{4\gamma}x^\circ\right)^{1/n}, \qquad \rho^\circ \approx 1 + \left(\frac{3n}{4\gamma}x^\circ\right)^{1/n} \\ h^\circ \approx \left(\frac{3n}{4\gamma}x^\circ\right)^{1/n}, \qquad p^\circ \approx \frac{\gamma-1}{\gamma}\left(\frac{3n}{4\gamma}x^\circ\right)^{1/n} \end{gathered} \tag{1.7}$$

In this manner, the behavior of the

solutions appear to be singular in the vicinity of the front in the general case. These results can well be compared with the analogous conclusions concerning the behavior of the solutions of nonlinear equations of heat flow in an infinite medium when in the initial state the medium has zero temperature [7].

Let us now clarify the physical meaning of the quantity l, which was introduced as a characteristic length. It is always proportional to the molecular mean free path evaluated behind a normal shock as $x \to +\infty$. In fact, the mean free path can be expressed in terms of macroscopic quantities [8]

$$l^\circ \approx \sqrt{\frac{\pi}{2}}\,\frac{\mu}{\sqrt{p\rho}}$$

Hence, utilizing (1.1), (1.2), (1.3), and substituting the values of the functions now known, we find

$$l = \frac{(\gamma+1)^{2n}}{2^{n-1}\gamma^n\sqrt{(\gamma-1)\pi}}\,l^\circ \qquad (\gamma = 1.4,\ l \approx 3.66 l^\circ \text{ when } n = 1) \tag{1.8}$$

2. The flow behind a body. Let us now turn to the study of the steady hypersonic flow of a viscous heat-conducting gas behind a body of finite dimensions. We shall restrict ourselves to the cases of plane flows and linear variations of viscosity with temperature (enthalpy) $\mu = Ch$. Again, we introduce dimensionless independent variables and dimensionless unknown functions

$$\begin{gathered} x^\circ = \frac{x}{l}, \qquad y^\circ = \frac{y}{l}, \qquad u^\circ = \frac{u}{V_\infty}, \qquad v^\circ = \frac{v}{V_\infty}, \qquad p^\circ = \frac{p}{\rho_\infty V_\infty^2} \\ \rho^\circ = \frac{\rho}{\rho_\infty}, \qquad h^\circ = \frac{h}{V_\infty^2} \qquad \left(l = \frac{CV_\infty}{\rho_\infty}\right) \end{gathered} \tag{2.1}$$

The equations of motion, continuity, heat flux, and state then have the form

$$\begin{gathered} \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{4}{3}h\frac{\partial u}{\partial x} - \frac{2}{3}h\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial y} + h\frac{\partial v}{\partial x}\right) \\ \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{4}{3}h\frac{\partial v}{\partial y} - \frac{2}{3}h\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial y} + h\frac{\partial v}{\partial x}\right) \\ u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \qquad p = \frac{\gamma-1}{\gamma}\rho h \\ \rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \frac{1}{\sigma}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{1}{\sigma}\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) + \\ + 2h\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + h\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 - \frac{2}{3}h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 \end{gathered} \tag{2.2}$$

Here and elsewhere we drop the superscript $^{\circ}$ in the designation of dimensionless quantities.

Let us focus on the nature of the flow far downstream from the body $x \gg L$, where L is a characteristic dimension of the body*. Since in the particular case of flow with $M_\infty \to \infty$ the disturbed region is circumscribed by a sharp front, let us first investigate the character of the flow near the front, i.e. the structure of the shock wave for $x \gg L$. The inclination τ of the front relative to the undisturbed stream direction can be considered small, when x is large (Fig. 3).

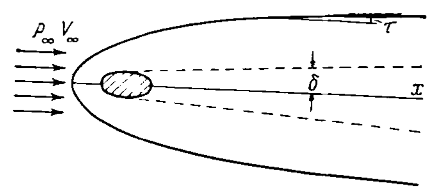

Fig. 3.

From the results of Section 1, it follows that for $\tau \ll 1$, the following assessments of magnitudes are valid in the neighborhood of the front

$$y \sim \tau x, \quad u \sim 1, \quad v \sim \tau, \quad \rho \sim 1, \quad p \sim \tau^2, \quad h \sim \tau^2 \tag{2.3}$$

For the increments in these quantities we have

$$\Delta x \sim 1, \ \Delta y \sim \tau, \ \Delta u \sim \tau^2, \ \Delta v \sim \tau, \ \Delta \rho \sim 1, \ \Delta p \sim \tau^2, \ \Delta h \sim \tau^2 \tag{2.4}$$

If we estimate the order of magnitudes of the terms in (2.2) on the basis of (2.3) and (2.4), and if we keep only the dominant terms, we arrive at approximate systems of equations:

$$\rho\left(\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial v}{\partial x}\right) - \frac{2}{3}\frac{\partial}{\partial x}\left(h\frac{\partial v}{\partial y}\right) \tag{2.5}$$

$$\rho\left(\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \frac{4}{3}\frac{\partial}{\partial y}\left(h\frac{\partial v}{\partial y}\right), \qquad \frac{\partial \rho}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

$$\rho\left(\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = \frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \frac{1}{\sigma}\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) + \frac{4}{3}h\left(\frac{\partial v}{\partial y}\right)^2, \qquad p = \frac{\gamma - 1}{\gamma}\rho h \tag{2.6}$$

The system of equations (2.6) is independent of (2.5) and, as is easily verified by changing x into the time variable t, is identical with the full system of equations of a uniform unsteady flow of a viscous

* Naturally, the condition $l \ll L$ must also be satisfied, so that we may apply the equations corresponding to a continuous medium.

heat-conducting gas. In this manner, we have established a complete equivalence between the flow in the shock wave at large distances downstream of a body and the uniform unsteady flow of a gas in the neighborhood of a front which moves according to the relation $dY/dt = r(t)$.

We note that the ratio of the terms neglected in (2.5) and (2.6) relative to the terms kept is of the order r^2; this determines the relative error of the results.

Let us now turn to the study of the flow of the inner region of the wake. The lateral dimensions of this region are on the order of $\Delta y \sim y \sim rx$. In this region, therefore, we can neglect not only the terms dropped from (2.2) in the derivation of the system (2.5), (2.6), but also all the viscous and heat-conductive terms, the ratio of which, with respect to the inertial terms, is clearly on the order of $1/x$. However, this state of affairs does not occur in the vicinity of the x-axis, where the role of viscosity and heat-conductivity is seen to be rather important. In fact, if we equate, as is usual, the magnitudes of the viscous and inertial terms in the first of Equations (2.2), we find that the influence of viscosity cannot be neglected near the x-axis up to a lateral distance of the order

$$\delta \sim \tau \sqrt{x} \tag{2.7}$$

The magnitude of the lateral velocity component in this flow region is clearly of the order

$$v \sim \frac{\delta}{x} \sim \frac{\tau}{\sqrt{x}} \tag{2.8}$$

The preceding estimates allow the simplification of the equations (2.2) in this inner region to the form

$$\begin{gathered}\rho\left(\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial y}\right) \\ \frac{\partial p}{\partial y} = 0, \quad \frac{\partial \rho}{\partial x} + \frac{\partial \rho v}{\partial y} = 0\end{gathered} \tag{2.9}$$

$$\rho\left(\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = \frac{\partial p}{\partial x} + \frac{1}{\sigma}\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right), \qquad p = \frac{\gamma - 1}{\gamma}\rho h \tag{2.10}$$

The ratio of the neglected terms in this equation to those retained is of the order r^2.

We shall not investigate here the question of integration of the resulting system of approximate equations (2.10). For later developments, the essential feature is the fact that the system (2.10) does not contain any terms which were left out in the derivation of Equations (2.6). Hence, Equations (2.6) can be viewed as characterizing the asymptotic behavior of the whole flow field at large distances downstream of the body. Therefore, the previously noted analogy with the unsteady uniform flow is also valid for the whole flow field in the wake. In physical terms, this means that at large distances downstream of a body which moves in a viscous heat-conducting gas with a very large supersonic speed, the "law of plane sections"* holds, in a manner similar to that for the case of inviscid and nonconducting gas [9]. In the central region of the wake, near the x-axis, the main influence on the transverse flow is exerted by heat conduction. That, of course, does not mean that the role of viscosity here is altogether unimportant; in accordance with (2.9), viscosity exerts the dominant influence on the variation of the x-component of velocity.

3. The aerodynamic drag of the body. For the complete specification of the flow field at large downstream distances it is necessary to establish its dependence on the aerodynamic resistance of the body and on the net heat flux across the boundary. This is not a difficult task if we utilize the demonstrated principle of equivalence between the flow in the wake and the one-dimensional unsteady flow of the gas. In this analogy, the corresponding unsteady flow of the gas is clearly iso-energetic, i.e. occurs as a result of a strong plane explosion. The energy E of the explosive charge, per unit area, needs to be equated to the sum of the aerodynamic drag X of the body, and the net integrated heat flow Q, per unit time, across the body boundaries:

$$E = X + Q \tag{3.1}$$

(The magnitudes E, X and Q are expressed in mechanical units and are dimensionless when referred to $\rho_\infty V_\infty^2 l$. The detailed derivation of the formula is given in Appendix B.)

On the other hand, the total energy of the unsteady gas flow at time t can be expressed as an integral of the internal and kinetic energies of the disturbed fluid elements:

$$E = 2\int_0^{Y(t)} \rho\left(\frac{h}{\gamma} + \frac{v^2}{2}\right) dy, \qquad Y(t) = \int_0^t \tau(t)\, dt \tag{3.2}$$

Here $Y(t)$ specifies the propagation of the shock front.

In this manner, the known gasdynamic analogy between hypersonic flows at large distances from the body (large compared to the characteristic

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* *Translator's Note:* See [1], p. 36.

dimensions of the body) and the one-dimensional unsteady flow behind a strong shock [9] is generalized to the case of the flow of a viscous heat-conducting gas in the particular case of an infinitely large Mach number. It is important to emphasize that the flow in the wake does not depend at all on the nature of the aerodynamic drag, i.e. on the relative contributions due to pressure and due to friction. Thus, for instance, the flow field at large distances downstream from two insulated flat plates ($Q = 0$), generating equal drag, would be identical even if one of the plates were aligned with the flow and the other normal to it.

These results permit a closer assessment of the behavior of the viscous wake near the axis. Let us use the well-known law of propagation of plane strong shock waves in inviscid gases:

$$Y(t) \sim t^{2/3} \tag{3.3}$$

as an approximation to the propagation in the case at hand. Then, we obtain

$$\tau \sim x^{-1/3} \tag{3.4}$$

for the characteristic inclination of the front for large x-values. Hence, with the aid of (2.7) we find that the viscous part of the wake spreads as

$$\delta \sim x^{1/6} \tag{3.5}$$

Finally, we note that the system (2.6) and the boundary conditions on the front ($\rho = 1$, $p = h = v = 0$) are invariant under the transformations

$$x = \xi, \quad y = a\eta, \quad v = av_1, \quad p = a^2 p_1, \quad \rho = \rho_1, \quad h = a^2 h_1 \tag{3.6}$$

With the following choice of the constant in (3.6):

$$a = E^{1/3} \tag{3.7}$$

the constancy of the total energy in the disturbed region states:

$$2 \int_0^{\eta_1(\xi)} p_1 \left(\frac{h_1}{\gamma} + \frac{v_1^2}{2} \right) d\eta = 1 \tag{3.8}$$

The relations (3.6) and (3.7) determine the law of similarity for flows with different values of energy E. In accordance with this law, the width of the disturbed region and the magnitude of the velocity of the flow vary as $E^{1/3}$, while the pressure and temperature vary as $E^{2/3}$.

4. Self-similar motions of a viscous heat-conducting gas. Should the temperature of the gas in the undisturbed region be zero, the only thermodynamic parameters determining the state of the gas in that region would be the density ρ_∞. Another basic dimensional parameter is given by the constant C in the dependence of viscosity on enthalpy (1.2). The dimensions of these quantities are

$$[\rho_\infty] = \frac{M}{L^3}, \qquad [C] = \frac{MT^{2n-1}}{L^{2n+1}} \tag{4.1}$$

where L, T and M stand for the unit measures of length, time and mass, respectively. If, amongst the other characteristic parameters of a given problem, no combination can be found with dimensions independent of ρ_∞ and C, then the problem is self-similar, as is well known [4]. We focus on the dimensions of the ratio ρ_∞ / C, so as to eliminate the units of mass

$$\left[\frac{\rho_\infty}{C}\right] = \frac{L^{2n-2}}{T^{2n-1}} \tag{4.2}$$

It follows that the following cases will have self-similar character:

a) uniformly accelerated motion of an insulated body, which has no characteristic length (infinite flat plate, cone, wedge), in the case $n = 3/2$, since then the ratio (4.2) has the dimensions of the acceleration $[a] = LT^{-2}$;

b) rotation of insulated axisymmetric conical bodies with $n = 1$, since then (4.2) has the dimensions of an angular velocity $[\Omega] = T^{-1}$;

c) point explosion with spherical symmetry, with $n = 1/6$, since then (4.2) has the same dimensions as

$$\left[\frac{\rho_\infty}{E}\right]^{1/3} = \frac{L^{-5/3}}{T^{-2/3}}$$

where E represents the energy of the explosion.

Let us investigate the first two cases of body motion in more detail. The case of the self-similar solution for the strong explosion has been carried out by Bam-Zelikovich [4].

a) Uniformly accelerated motion of a flat plate. Let us study the simplest of the problems of the first type, namely, the motion of a viscous heat-conducting gas induced by an infinite flat plate which starts accelerating uniformly from rest at a given instant. The acceleration vector is inclined at an angle a relative to the surface of the flat plate (Fig. 4). The x-axis of a Cartesian system of coordinates is taken coincident with the flat plate at the initial instant. Clearly, the motion depends only on two independent variables, y and t. Since, among the characteristic parameters of the problem there are none with dimensions of length and time, the only possible dimensionless combination of the independent variables and of these parameters is

$$\eta = \frac{2y}{at^2} \tag{4.3}$$

The unknown functions of the problem are the components of the velocity u and v, the pressure p, the density ρ, and the enthalpy h. It can be seen that their only possible functional ties with the independent variables are of the form

$$u = atV(\eta),\quad v = atV(\eta) \tag{4.4}$$

$$p = \rho_\infty a^2 t^2 P(\eta),\quad \rho = \rho_\infty R(\eta),\quad h = a^2 t^2\, H(\eta)$$

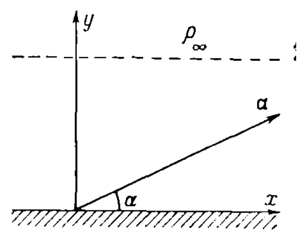

Fig. 4.

According to the preceding analysis, the dependence of viscosity on enthalpy takes the form

$$\mu = Ch^{3/2} = Ca^3 t^3 H^{3/2}(\eta) \tag{4.5}$$

Substitution of Equations (4.3), (4.4) and (4.5) into the full set of the equations for the motion of a viscous heat-conducting gas leads to a system of ordinary differential equations for the dimensionless functions U, V, P, R, H, which are not given here in detail. The dimensionless parameter of the problem is clearly

$$k = \frac{\rho_\infty}{aC}$$

It characterizes the effect of viscosity and heat-conductivity of the gas and resembles the Reynolds number.

Let us now turn to the boundary conditions of the problem. At the insulated plate we must satisfy the conditions of no slip and of zero heat-flux. We can easily verify that these conditions, expressed in terms of the dimensionless variables, are

$$U(1) = \cos\ \alpha,\quad V(1) = \sin\ \alpha,\quad H'(1) = 0 \tag{4.6}$$

In the undisturbed region the temperature is zero. As has been shown, this region is therefore separated from the region of disturbed motion by a front* which obviously propagates according to the similarity law

* The mathematical necessity of the existence of the front can be established by assuming the opposite; the extension of the asymptotic solutions to infinity then leads to a contradiction.

$$Y(t) = \eta_0 \frac{at^2}{2} \qquad (4.7)$$

Here η_0 is a constant which must be determined. Clearly, the boundary conditions at the front have the form

$$U(\eta_0) = V(\eta_0) = P(\eta_0) = H(\eta_0) = 0, \qquad R(\eta_0) = 1 \qquad (4.8)$$

The conditions (4.6) and (4.8) must fully determine the solution of the boundary-value problem, including the specification of the constant η_0.

b) The motion of a viscous heat-conducting gas induced by the rotation of a conical surface. Let us examine the self-similar motion of the second type associated with a uniform rotation of an axisymmetric conical surface around its axis. Let us place the origin of the spherical coordinates r, θ, ϕ at the vertex of the cone (Fig. 5) and make its axis of symmetry coincide with $\theta = 0$. We shall again assume that the rotation with constant angular velocity Ω begins from rest at time $t = 0$ and that the temperature of the undisturbed medium is zero. Clearly, the motion in question can depend only on the three independent variables r, θ and t. Since the only characteristic parameter of the problem, which does not contain the dimension of mass, is the angular velocity, the dependence on the variable r can be found in an explicit form. For the dimensionless time variable let us take

$$\tau = \Omega t \qquad (4.9)$$

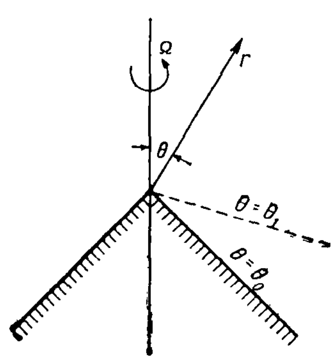

Fig. 5.

The unknown functions of the problem are the components of the velocity vector V_r, V_θ and V_ϕ, the pressure p, the density ρ and the gas enthalpy h. We can easily see that the only possible form of the dependence of these functions on the independent variables is

$$V_r = \Omega r U(\theta, \tau), \quad V_\theta = \Omega r V(\theta, \tau) \quad (4.10)$$
$$V_\varphi = \Omega r\, W(\theta, \tau), \qquad p = \rho_\infty \Omega^2 r^2\, P(\theta, \tau)$$
$$\rho = \rho_\infty R(\theta, \tau), \qquad h = \Omega^2 r^2 H(\theta, \tau)$$

As has been established earlier, the viscosity will depend linearly on enthalpy

$$\mu = Ch = C\Omega^2 r^2 H(\theta\, \tau) \qquad (4.11)$$

Upon substitution of (4.9), (4.10) and (4.11) into the equations of motion of a viscous heat-conducting gas, we obtain a system of partial differential equations in two independent variables for the dimensionless functions (4.10). The conditions at the boundary $\theta = \theta_0$, which is heat-insulated, are

$$U = V = \frac{\partial H}{\partial \theta} = 0, \qquad W = \sin\,\theta_0 \tag{4.12}$$

It is also necessary to satisfy conditions on the front which moves according to the law $\theta = \theta_1(r)$, yet to be determined. These conditions are

$$U = V = W = P = H = 0, \qquad R = 1 \tag{4.13}$$

The system of boundary conditions (4.12) and (4.13) must fully determine the solution of the problem, including the unknown function $\theta_1(r)$.

We note that from (4.13) it follows that the front which separates the disturbed region from that of the undisturbed gas spreads as a conical surface with a vertex at the vertex of the rotating cone. Furthermore, from the form of the solutions (4.10) it follows that the velocity components grow linearly with r, the pressure and enthalpy as r^2, while the density remains constant along any ray issuing from the origin.

We have examined some special problems associated with the motion of a viscous heat-conducting gas at very high supersonic speeds. However, the results permit some general deductions concerning the nature of the problem of flow around arbitrary bodies as $M_\infty \to \infty$. The most important feature of any flow with infinite Mach number is the appearance of a frontal surface, which separates the disturbed flow field from the region of the uniform stream. Consequently, the conditions of asymptotic decay of all disturbances at infinity are always replaced by boundary conditions on the surface of the front. The behavior of the solution in the vicinity of this front is singular and can be obtained relatively simply by examining the equations of motion in a small neighborhood of the front. In this process, any segment of the front can be considered as flat and its velocity (in the direction of its normal) as constant during a small interval of time. The reduction of the problem of flow around a body with $M_\infty \to \infty$ to a boundary-value problem in a finite domain facilitates its solution by approximate methods (as in the case of flows of ideal gases around the body).

APPENDIX A. The integration of the system of equations (1.4) in the special case of Prandtl number 3/4 can be carried out relatively simply. The first of these equations together with the boundary condition (1.5) yields

$$\rho^\circ u^\circ = 1 \tag{A.1}$$

Integrating the second equation we find

$$u^\circ + p^\circ = \frac{4}{3} h^{\circ n} \frac{du^\circ}{dx^\circ} + 1 \tag{A.2}$$

The third equation can be rewritten with the aid of the second in the form

$$\frac{d}{dx^\circ}\left(h^\circ + \frac{u^{\circ 2}}{2}\right) = \frac{4}{3}\frac{d}{dx^\circ}\left[h^{\circ n}\frac{d}{dx^\circ}\left(h^\circ + \frac{u^{\circ 2}}{2}\right)\right] \tag{A.3}$$

Its only particular solution which satisfies (1.5) is

$$h^\circ + \frac{u^{\circ 2}}{2} = \frac{1}{2} \tag{A.4}$$

Substituting into (A.2) the expression (A.4) for h° and

$$p^\circ = \frac{\gamma - 1}{\gamma}\rho^\circ h^\circ = \frac{\gamma - 1}{\gamma}\frac{h^\circ}{u^\circ} = \frac{\gamma - 1}{2\gamma}\frac{1 - u^{\circ 2}}{u^\circ} \tag{A.5}$$

we obtain an equation in u° alone:

$$\frac{2^{2-n}}{3}(1 - u^{\circ 2})^n u^\circ \frac{du^\circ}{dx^\circ} + \frac{\gamma + 1}{2\gamma}(1 - u^\circ)\left(u^\circ - \frac{\gamma - 1}{\gamma + 1}\right) = 0 \tag{A.6}$$

Its integration in the case $n = 1$ is elementary and leads to (1.6). In the case of an arbitrary $n > 0$, the quantity u° near the front can be expressed as

$$u^\circ \approx 1 - ax^\alpha \tag{A.7}$$

Substituting this expression into (A.6) and keeping the dominant terms, we find

$$\alpha = \frac{1}{n}, \qquad a\left(\frac{3}{4}\frac{n}{\gamma}\right)^{1/n} \tag{A.8}$$

Equation (1.7) then follows.

APPENDIX B. The full equations of motion of a viscous heat-conducting gas (2.2) can be written using the divergence operator and then expressed in the form of the following integrals over closed contour surfaces:

$$\oint\left(h\frac{\partial u}{\partial y} + h\frac{\partial v}{\partial x} - \rho uv\right)dx + \left(p + \rho u^2 - \frac{4}{3}h\frac{\partial u}{\partial x} + \frac{2}{3}h\frac{\partial v}{\partial y}\right)dy = 0$$
$$\oint\left(\frac{4}{3}h\frac{\partial v}{\partial y} - \frac{2}{3}h\frac{\partial u}{\partial x} - p - \rho v^2\right)dx + \left(\rho uv - h\frac{\partial u}{\partial y} - h\frac{\partial v}{\partial y}\right)dy = 0$$
$$\oint - \rho v\, dx + \rho u\, dy = 0 \tag{B.1}$$

$$\oint\left(-\rho v h-\rho v\frac{u^2}{2}-\rho v\frac{v^2}{2}+\frac{h}{\sigma}\frac{\partial h}{\partial y}+hu\frac{\partial u}{\partial y}+hu\frac{\partial v}{\partial x}+\frac{4}{3}hv\frac{\partial v}{\partial y}+\frac{2}{3}hv\frac{\partial u}{\partial x}\right)dx+$$
$$+\left(\rho u h+\rho u\frac{u^2}{2}+\rho u\frac{v^2}{2}-\frac{h}{\sigma}\frac{\partial h}{\partial x}-\frac{4}{3}hu\frac{\partial u}{\partial x}+\frac{2}{3}hu\frac{\partial v}{\partial y}-hv\frac{\partial u}{\partial y}-hv\frac{\partial v}{\partial x}\right)dy=0$$

Let us choose for the contour of integration the contour *ABCDEF* (Fig. 6), with *A* at the vertex of the front *AE* and with the closing section *DE* located far downstream of the body *BC*. The full resistance of the body, *X*, is given by the integral of the normal and tangential stresses projected in the *x*-direction:

$$X=2\int_B^C\left(h\frac{\partial u}{\partial y}+h\frac{\partial v}{\partial x}\right)dx+\left(p-\frac{4}{3}h\frac{\partial u}{\partial x}+\frac{2}{3}h\frac{\partial v}{\partial y}\right)dy \tag{B.2}$$

and the net flux of heat across the boundaries of the body by the integral

$$Q=2\int_B^C\frac{h}{\sigma}\left(\frac{\partial h}{\partial y}dx-\frac{\partial h}{\partial x}dy\right) \tag{B.3}$$

Taking into account the conditions at the surface of the body and in the undisturbed flow, we obtain

$$X=-2\int_0^Y\left(p+\rho u^2-\frac{4}{3}h\frac{\partial u}{\partial x}+\frac{2}{3}h\frac{\partial v}{\partial y}\right)dy+2\int_0^Y dy \tag{B.4}$$

$$Q=2\int_0^Y\left(\rho u h+\rho u\frac{u^2}{2}+\rho u\frac{v^2}{2}-\frac{h}{\sigma}\frac{\partial h}{\partial x}-\frac{4}{3}hu\frac{\partial u}{\partial x}+\right.$$
$$\left.+\frac{2}{3}hu\frac{\partial v}{\partial y}-hv\frac{\partial u}{\partial y}-hv\frac{\partial v}{\partial x}\right)dy-\int_0^Y dy \tag{B.5}$$

Let us rewrite the condition of conservation of mass in the form

$$\int_0^Y\rho u\,dy=\int_0^Y dy \tag{B.6}$$

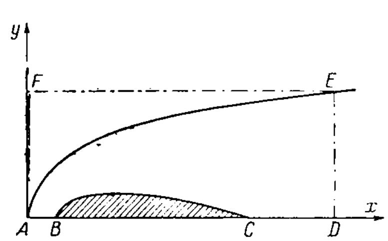

Fig. 6.

When, in accordance with the estimates of Section 2, we neglect in these equations the terms of higher order, we find

$$X+Q=2\int_0^{Y(x)}\rho\left(\frac{h}{\gamma}+\frac{v^2}{2}\right)dy \tag{B.7}$$

On the other hand, the right-hand side of (B.7) is easily seen to represent the total energy of the equivalent uniform unsteady motion of the gas. This validates Equation (3.1). We note that the ratio of the terms neglected in (B.4), (B.5) and (B.6) to those which were kept is of the order r^2.

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